Published by Institute of Physics Publishing for SISSA

Received: *September 7, 2007*
Accepted: *September 15, 2007*
Published: *September 19, 2007*

Real analytic solutions for marginal deformations in open superstring field theory

Yuji Okawa

DESY Theory Group,
Notkestrasse 85, 22607 Hamburg, Germany
yuji.okawa@desy.de

Abstract: We construct analytic solutions for marginal deformations satisfying the reality condition in open superstring field theory formulated by Berkovits when operator products made of the marginal operator and the associated superconformal primary field are regular. Our strategy is based on the recent observation by Erler that the problem of finding solutions for marginal deformations in open superstring field theory can be reduced to a problem in the bosonic theory of finding a finite gauge parameter for a certain pure-gauge configuration labeled by the parameter of the marginal deformation. We find a gauge transformation generated by a real gauge parameter which infinitesimally changes the deformation parameter and construct a finite gauge parameter by its path-ordered exponential. The resulting solution satisfies the reality condition by construction.

Keywords: String Field Theory, Superstrings and Heterotic Strings.

Contents

1. Introduction

Analytic methods in open bosonic string field theory [1][1] triggered by Schnabl's construction of an analytic solution for tachyon condensation [6] and further developed in [7 – 21] have recently been extended to open superstring field theory formulated by Berkovits [22], and analytic solutions for marginal deformations were constructed in [23, 24].[2] The solutions are surprisingly simple and very similar to those in open bosonic string field theory constructed in [20, 21]. However, the reality condition on the open superstring field was not satisfied. While we expect that the solution in [23, 24] is equivalent to a real one by a gauge transformation, it is desirable to find an analytic solution satisfying the reality condition. In this paper we explicitly construct a real analytic solution.

The equation of motion in open superstring field theory [22] is

$$\eta_0(e^{-\Phi} Q_B e^{\Phi}) = 0, \tag{1.1}$$

where Φ is the open superstring field and Q_B is the BRST operator. The superghost sector is described by η, ξ, and ϕ [40, 41], and η_0 is the zero mode of η. All the string products in this paper are defined by the star product introduced in [1]. For any marginal deformation of the boundary conformal field theory (CFT) for the open superstring, there is an associated superconformal primary field $V_{1/2}$ of dimension 1/2, and the marginal operator V_1 of dimension 1 is the supersymmetry transformation of $V_{1/2}$. In open superstring field theory [22], the solution to the linearized equation of motion associated with the marginal deformation is given by the Grassmann-even state X corresponding to the

[1] See [2 – 5] for reviews on string field theory.

[2] For earlier study of marginal deformations in string field theory and related work, see [25 – 39].

operator $V(0) = c\xi e^{-\phi} V_{1/2}(0)$ in the state-operator mapping. When the deformation is exactly marginal, we expect a solution to (1.1) of the following form:

$$\Phi_\lambda = \sum_{n=1}^{\infty} \lambda^n \Phi^{(n)} \tag{1.2}$$

with $\Phi^{(1)} = X$, where λ is the deformation parameter. The goal of the paper is to construct $\Phi^{(n)}$ satisfying the reality condition when operator products made of V_1 and $V_{1/2}$ are regular.

In [23] Erler proposed to solve the following equation:

$$e^{-\Phi} Q_B e^{\Phi} = \Psi_\lambda, \tag{1.3}$$

where Ψ_λ is obtained from the solution for marginal deformations in open bosonic string field theory constructed in [20, 21] by replacing the state corresponding to $cV_b(0)$ for the bosonic string with the state $Q_B X$ for the superstring, where V_b is the marginal operator in the bosonic theory. The state Ψ_λ satisfies the equation of motion in open bosonic string field theory,

$$Q_B \Psi_\lambda + \Psi_\lambda^2 = 0, \tag{1.4}$$

and to linear order in λ it reduces to

$$\Psi_\lambda = \lambda Q_B X + O(\lambda^2). \tag{1.5}$$

Thus Ψ_λ is a pure-gauge solution generated by $Q_B X$, and we expect a solution to (1.3) of the form

$$\Phi = \lambda X + O(\lambda^2). \tag{1.6}$$

Furthermore, Ψ_λ is annihilated by η_0 because the state X satisfies the linearized equation of motion $\eta_0 Q_B X = 0$. Therefore, the solution to (1.3) solves the equation of motion in open superstring field theory (1.1), and the problem of solving the superstring theory has been reduced to a problem in the bosonic theory. A simple solution to (1.3) was obtained in [23], but the reality condition on the open superstring field was not satisfied. The same solution was also obtained in [24] by a different approach.

Let us now consider the equation obtained from (1.3) by taking a derivative with respect to λ. Since the left-hand side of (1.3) takes the form of a pure-gauge configuration with respect to the gauge transformation in the bosonic theory, its infinitesimal change must be written as an infinitesimal gauge transformation generated by some gauge parameter which we call $G(\lambda)$:

$$Q_B G(\lambda) + [\Psi_\lambda, G(\lambda)] = \Psi'_\lambda, \qquad \Psi'_\lambda \equiv \frac{d}{d\lambda} \Psi_\lambda. \tag{1.7}$$

Then a solution to (1.3) can be constructed by a path-ordered exponential of $G(\lambda)$ as

$$e^{\Phi_\lambda} = \mathrm{Pexp}\Big[\int_0^\lambda d\lambda' G(\lambda') \Big], \tag{1.8}$$

or

$$\Phi_\lambda = \ln \mathrm{Pexp}\biggl[\int_0^\lambda d\lambda' G(\lambda')\biggr]. \tag{1.9}$$

If $G(\lambda)$ satisfies the reality condition, the solution Φ_λ also satisfies the reality condition by construction. This is our strategy for constructing a real solution in open superstring field theory. It turns out that it is easy to find a real solution to (1.7).

After we completed the construction of solutions satisfying the reality condition, we learned that T. Erler independently constructed analytic solutions satisfying the reality condition by a different approach. His solutions were presented in the second version of [23].

2. Pure-gauge string field

Let us begin with describing Ψ_λ in (1.3). It is obtained from the solution for marginal deformations in open bosonic string field theory constructed in [20, 21] by replacing cV_b in the bosonic theory with the BRST transformation of $V = c\xi e^{-\phi}V_{1/2}$ for the superstring. This section largely overlaps with section 2 of [24], where the solution in open bosonic string field theory was reviewed. The string field Ψ_λ is defined by an expansion with respect to λ as follows:

$$\Psi_\lambda = \sum_{n=1}^{\infty} \lambda^n \Psi^{(n)}. \tag{2.1}$$

The BPZ inner product $\langle \varphi, \Psi^{(n)} \rangle$ with a state φ in the Fock space is given by

$$\begin{aligned}\langle \varphi, \Psi^{(n)} \rangle = \int_0^1 dt_1 \int_0^1 dt_2 \ldots \int_0^1 dt_{n-1} \langle f \circ \varphi(0) U(1) \mathcal{B} U(1+t_1) \mathcal{B} U(1+t_1+t_2) \ldots \\ \times \mathcal{B} U(1+t_1+t_2+\ldots+t_{n-1}) \rangle_{\mathcal{W}_{1+t_1+t_2+\ldots+t_{n-1}}},\end{aligned} \tag{2.2}$$

where U is the BRST transformation of V:

$$U(z) = Q_B \cdot V(z), \qquad V(z) = c\xi e^{-\phi} V_{1/2}(z). \tag{2.3}$$

We follow the notation used in [7, 14, 21]. In particular, see the beginning of section 2 of [7] for the relation to the notation used in [6]. Here and in what follows we use φ to denote a generic state in the Fock space and $\varphi(0)$ to denote its corresponding operator in the state-operator mapping. We use the doubling trick in calculating CFT correlation functions. As in [14], we define the oriented straight lines $V_\alpha^\pm$ by

$$\begin{aligned} V_\alpha^\pm &= \Bigl\{ z \Big| \mathrm{Re}(z) = \pm \frac{1}{2}(1+\alpha) \Bigr\}, \\ \text{orientation} &: \pm \frac{1}{2}(1+\alpha) - i\infty \to \pm \frac{1}{2}(1+\alpha) + i\infty, \end{aligned} \tag{2.4}$$

and the surface $\mathcal{W}_\alpha$ can be represented as the region between V_0^- and $V_{2\alpha}^+$, where V_0^- and $V_{2\alpha}^+$ are identified by translation. The function $f(z)$ is

$$f(z) = \frac{2}{\pi} \arctan z, \tag{2.5}$$

and $f \circ \varphi(z)$ denotes the conformal transformation of $\varphi(z)$ by the map $f(z)$. The operator $\mathcal{B}$ is defined by

$$\mathcal{B} = \int \frac{dz}{2\pi i} b(z), \tag{2.6}$$

and when $\mathcal{B}$ is located between two operators at t_1 and t_2 with $1/2 < t_1 < t_2$, the contour of the integral can be taken to be $-V_\alpha^+$ with $2t_1 - 1 < \alpha < 2t_2 - 1$. The anticommutation relation of $\mathcal{B}$ and $c(z)$ is

$$\{\mathcal{B}, c(z)\} = 1, \tag{2.7}$$

and $\mathcal{B}^2 = 0$.

The state $\Psi^{(n)}$ can be written more compactly as

$$\langle \varphi, \Psi^{(n)} \rangle = \int d^{n-1}t \Big\langle f \circ \varphi(0) \prod_{i=0}^{n-2} \Big[U(1+\ell_i)\mathcal{B} \Big] U(1+\ell_{n-1}) \Big\rangle_{\mathcal{W}_{1+\ell_{n-1}}}, \tag{2.8}$$

where

$$\int d^{n-1}t \equiv \int_0^1 dt_1 \int_0^1 dt_2 \ldots \int_0^1 dt_{n-1}, \quad \ell_0 = 0, \quad \ell_i \equiv \sum_{k=1}^{i} t_k \quad \text{for} \quad i = 1, 2, 3, \ldots. \tag{2.9}$$

The state Ψ_λ can be represented as

$$\Psi_\lambda = \frac{1}{1 - \lambda(Q_B X)P} \lambda Q_B X, \tag{2.10}$$

where

$$\frac{1}{1 - \lambda(Q_B X)P} \equiv 1 + \sum_{n=1}^{\infty} [\lambda(Q_B X)P]^n. \tag{2.11}$$

The state X is described in the CFT language as

$$\langle \varphi, X \rangle = \langle f \circ \varphi(0) V(1) \rangle_{\mathcal{W}_1} = \langle f \circ \varphi(0) c\xi e^{-\phi} V_{1/2}(1) \rangle_{\mathcal{W}_1}, \tag{2.12}$$

and the state $Q_B X$ is

$$\langle \varphi, Q_B X \rangle = \langle f \circ \varphi(0) Q_B \cdot V(1) \rangle_{\mathcal{W}_1} = \langle f \circ \varphi(0) U(1) \rangle_{\mathcal{W}_1}. \tag{2.13}$$

The definition of P is a little involved.[3] It is defined when it appears as $\varphi_1 P \varphi_2$ between two states φ_1 and φ_2 in the Fock space. The string product $\varphi_1 P \varphi_2$ is given by

$$\langle \varphi, \varphi_1 P \varphi_2 \rangle = \int_0^1 dt \langle f \circ \varphi(0) f_1 \circ \varphi_1(0) \mathcal{B} f_{1+t} \circ \varphi_2(0) \rangle_{\mathcal{W}_{1+t}}, \tag{2.14}$$

where $\varphi_1(0)$ and $\varphi_2(0)$ are the operators corresponding to the states φ_1 and φ_2, respectively. The map $f_a(z)$ is a combination of $f(z)$ and translation:

$$f_a(z) = \frac{2}{\pi} \arctan z + a. \tag{2.15}$$

[3]The state P corresponds to J_b of [24] in the bosonic case and to $\eta_0 J$ of [24] in the superstring case.

JHEP09(2007)082

The string product $\varphi_1 P \varphi_2$ is well defined if $f_1 \circ \varphi_1(0) \mathcal{B} f_{1+t} \circ \varphi_2(0)$ is regular in the limit $t \to 0$.

An important property of P is

$$\varphi_1 (Q_B P) \varphi_2 = \varphi_1 \varphi_2 \tag{2.16}$$

when $f_1 \circ \varphi_1(0) f_{1+t} \circ \varphi_2(0)$ vanishes in the limit $t \to 0$. This relation can be shown in the following way. Since the BRST transformation of $b(z)$ is the energy-momentum tensor $T(z)$, the inner product $\langle \varphi, \varphi_1 (Q_B P) \varphi_2 \rangle$ is given by

$$\langle \varphi, \varphi_1 (Q_B P) \varphi_2 \rangle = \int_0^1 dt \langle f \circ \varphi(0) f_1 \circ \varphi_1(0) \mathcal{L} f_{1+t} \circ \varphi_2(0) \rangle_{\mathcal{W}_{1+t}}, \tag{2.17}$$

where

$$\mathcal{L} = \int \frac{dz}{2\pi i} T(z), \tag{2.18}$$

and the contour of the integral is the same as that of $\mathcal{B}$. As discussed in [7], an insertion of $\mathcal{L}$ is equivalent to taking a derivative with respect to t. It is analogous to the relation $L_0 e^{-tL_0} = -\partial_t e^{-tL_0}$ in the standard strip coordinates, where L_0 is the zero mode of the energy-momentum tensor. We thus have

$$\begin{aligned} \langle \varphi, \varphi_1 (Q_B P) \varphi_2 \rangle &= \int_0^1 dt \partial_t \langle f \circ \varphi(0) f_1 \circ \varphi_1(0) f_{1+t} \circ \varphi_2(0) \rangle_{\mathcal{W}_{1+t}} \\ &= \langle f \circ \varphi(0) f_1 \circ \varphi_1(0) f_2 \circ \varphi_2(0) \rangle_{\mathcal{W}_2} = \langle \varphi, \varphi_1 \varphi_2 \rangle \end{aligned} \tag{2.19}$$

when $f_1 \circ \varphi_1(0) f_{1+t} \circ \varphi_2(0)$ vanishes in the limit $t \to 0$. This completes the proof of (2.16). In the language of [21], $\varphi_1 P \varphi_2$ is

$$\varphi_1 P \varphi_2 = \int_0^1 dt \varphi_1 e^{-(t-1)L_L^+} (-B_L^+) \varphi_2, \tag{2.20}$$

and the relation (2.16) follows from $\{Q_B, B_L^+\} = L_L^+$.

To summarize, when the regularity conditions we mentioned are satisfied, Ψ_λ is well defined, and we can safely use the relation

$$Q_B P = 1 \tag{2.21}$$

for the Grassmann-odd state P. It is then straightforward to calculate $Q_B \Psi_\lambda$, and the result is

$$Q_B \Psi_\lambda = -\frac{1}{1 - \lambda (Q_B X) P} \lambda (Q_B X) \frac{1}{1 - \lambda (Q_B X) P} \lambda Q_B X. \tag{2.22}$$

We have thus shown that Ψ_λ satisfies the equation of motion for the bosonic string:

$$Q_B \Psi_\lambda + \Psi_\lambda^2 = 0. \tag{2.23}$$

Another important property of Ψ_λ is that $\eta_0 \Psi_\lambda = 0$. It is easy to see that η_0 annihilates $\Psi^{(n)}$ in (2.2) because η and b anticommute and U is annihilated by η_0.

3. Solution

Let us now solve

$$Q_B G(\lambda) + [\Psi_\lambda, G(\lambda)] = \Psi'_\lambda. \tag{3.1}$$

The string field Ψ'_λ is given by

$$\Psi'_\lambda \equiv \frac{d}{d\lambda}\Psi_\lambda = \frac{1}{1-\lambda(Q_BX)P}(Q_BX)\frac{1}{1-\lambda P(Q_BX)}, \tag{3.2}$$

where

$$\frac{1}{1-\lambda(Q_BX)P} \equiv 1 + \sum_{n=1}^{\infty}[\lambda(Q_BX)P]^n, \quad \frac{1}{1-\lambda P(Q_BX)} \equiv 1 + \sum_{n=1}^{\infty}[\lambda P(Q_BX)]^n. \tag{3.3}$$

We look for a solution made of X, P, and Q_B. We assume for the moment that states involving P are well defined and that we can use the relation $Q_BP = 1$. We will discuss regularity conditions necessary for these assumptions later. A string field within this ansatz satisfies the reality condition if it is odd under the conjugation given by replacing $X \to -X$ and by reversing the order of string products. Signs from anticommuting Grassmann-odd string fields have to be taken care of in reversing the order of string products. For example, the state $\Psi^{(n)}$ is real because its conjugation is given by

$$\begin{aligned}&\Psi^{(n)} = [(Q_BX)P]^{n-1}Q_BX\\ &\to (-1)^{(2n-1)(n-1)}(-Q_BX)[P(-Q_BX)]^{n-1} = -[(Q_BX)P]^{n-1}Q_BX\end{aligned} \tag{3.4}$$

for any positive integer n. It is easy to find a perturbative solution to (3.1) by expanding the equation and $G(\lambda)$ in powers of λ. We find that the following state solves (3.1) to all orders in λ and satisfies the reality condition:

$$G(\lambda) = \frac{1}{1-\lambda(Q_BX)P}X\frac{1}{1-\lambda P(Q_BX)}. \tag{3.5}$$

It is easy to see that $G(\lambda)$ in (3.5) solves (3.1) from the following relations:

$$\begin{aligned}Q_B\frac{1}{1-\lambda(Q_BX)P} &= -\frac{1}{1-\lambda(Q_BX)P}\lambda(Q_BX)\frac{1}{1-\lambda(Q_BX)P} = -\Psi_\lambda\frac{1}{1-\lambda(Q_BX)P},\\ Q_B\frac{1}{1-\lambda P(Q_BX)} &= \frac{1}{1-\lambda P(Q_BX)}\lambda(Q_BX)\frac{1}{1-\lambda P(Q_BX)} = \frac{1}{1-\lambda P(Q_BX)}\Psi_\lambda.\end{aligned} \tag{3.6}$$

An explicit expression of $G(\lambda)$ in the CFT description is given by

$$\begin{aligned}\langle\varphi, G(\lambda)\rangle = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\lambda^{n+m}\int d^{n+m}t\Big\langle f\circ\varphi(0)\prod_{i=0}^{n-1}\Big[U(1+\ell_i)\mathcal{B}\Big]V(1+\ell_n)&\\ \times\prod_{j=n+1}^{n+m}\Big[\mathcal{B}U(1+\ell_j)\Big]\Big\rangle_{\mathcal{W}_{1+\ell_{n+m}}}&,\end{aligned} \tag{3.7}$$

with the understanding that

$$\prod_{i=0}^{-1}\Big[U(1+\ell_i)\mathcal{B}\Big]=1,\qquad \prod_{j=n+1}^{n}\Big[\mathcal{B}U(1+\ell_j)\Big]=1,\qquad \int d^0t=1. \tag{3.8}$$

Following the strategy outlined in the introduction, we construct a solution to the equation of motion (1.1) in open superstring field theory as follows:

$$e^{\Phi_\lambda}=\mathrm{Pexp}\bigg[\int_0^\lambda d\lambda' G(\lambda')\bigg], \tag{3.9}$$

or

$$\Phi_\lambda=\ln\mathrm{Pexp}\bigg[\int_0^\lambda d\lambda' G(\lambda')\bigg], \tag{3.10}$$

where our convention for the path-ordered exponential is

$$\begin{aligned}\mathrm{Pexp}\bigg[\int_a^b d\lambda' G(\lambda')\bigg]=1&+\int_a^b d\lambda_1 G(\lambda_1)+\int_a^b d\lambda_1\int_a^{\lambda_1} d\lambda_2 G(\lambda_2)G(\lambda_1)\\ &+\int_a^b d\lambda_1\int_a^{\lambda_1} d\lambda_2\int_a^{\lambda_2} d\lambda_3 G(\lambda_3)G(\lambda_2)G(\lambda_1)+\ldots.\end{aligned} \tag{3.11}$$

It can also be written as

$$\begin{aligned}\mathrm{Pexp}\bigg[\int_a^b d\lambda' G(\lambda')\bigg]=1&+\int_a^b d\lambda_1 G(\lambda_1)+\int_a^b d\lambda_1\int_{\lambda_1}^{b} d\lambda_2 G(\lambda_1)G(\lambda_2)\\ &+\int_a^b d\lambda_1\int_{\lambda_1}^{b} d\lambda_2\int_{\lambda_2}^{b} d\lambda_3 G(\lambda_1)G(\lambda_2)G(\lambda_3)+\ldots.\end{aligned} \tag{3.12}$$

The path-ordered exponential satisfies the differential equations given by

$$\begin{aligned}\frac{d}{db}\mathrm{Pexp}\bigg[\int_a^b d\lambda' G(\lambda')\bigg]&=\mathrm{Pexp}\bigg[\int_a^b d\lambda' G(\lambda')\bigg]G(b),\\ \frac{d}{da}\mathrm{Pexp}\bigg[\int_a^b d\lambda' G(\lambda')\bigg]&=-G(a)\mathrm{Pexp}\bigg[\int_a^b d\lambda' G(\lambda')\bigg],\end{aligned} \tag{3.13}$$

with the initial condition

$$\mathrm{Pexp}\bigg[\int_a^b d\lambda' G(\lambda')\bigg]\bigg|_{a=b}=1. \tag{3.14}$$

The string field $e^{-\Phi_\lambda}$ is given by

$$e^{-\Phi_\lambda}=\mathrm{Pexp}\bigg[\int_\lambda^0 d\lambda' G(\lambda')\bigg]. \tag{3.15}$$

It is straightforward to verify that (3.1) can be obtained from (1.3) with $\Phi=\Phi_\lambda$ in (3.10) by taking a derivative with respect to λ. The equation of motion is trivially satisfied when $\lambda=0$. Thus Φ_λ in (3.10) satisfies the equation of motion (1.1) to all orders in λ. This is the main result of this paper. We present the expansion of Φ_λ to $O(\lambda^3)$ in appendix A. While it is guaranteed that Φ_λ satisfies the reality condition by construction, we can

JHEP09(2007)082

explicitly confirm this. Since the conjugate of $G(\lambda)$ associated with the reality condition is $-G(\lambda)$, the conjugate of e^{Φ_λ} is $e^{-\Phi_\lambda}$, as can be seen using the formulas (3.11) and (3.12). Therefore, its logarithm Φ_λ satisfies the reality condition.

The analytic solution constructed in [23, 24] can also be written using a path-ordered exponential. Let us denote the solution in [23, 24] by $\widetilde{\Phi}_\lambda$. It is given by

$$e^{\widetilde{\Phi}_\lambda} = \frac{1}{1 - H_\lambda}, \tag{3.16}$$

where

$$H_\lambda = \frac{1}{1 - \lambda (Q_B X) P} \lambda X. \tag{3.17}$$

It is easy to calculate $Q_B H_\lambda$ and show that

$$e^{-\widetilde{\Phi}_\lambda} Q_B e^{\widetilde{\Phi}_\lambda} = (Q_B H_\lambda) \frac{1}{1 - H_\lambda} = \Psi_\lambda. \tag{3.18}$$

Thus $\widetilde{\Phi}_\lambda$ solves the equation of motion (1.1). Since

$$\frac{d}{d\lambda} e^{\widetilde{\Phi}_\lambda} = \frac{1}{1 - H_\lambda} H'_\lambda \frac{1}{1 - H_\lambda} = e^{\widetilde{\Phi}_\lambda} H'_\lambda \frac{1}{1 - H_\lambda}, \tag{3.19}$$

where

$$H'_\lambda \equiv \frac{d}{d\lambda} H_\lambda, \tag{3.20}$$

and $e^{\widetilde{\Phi}_\lambda} = 1$ at $\lambda = 0$, $e^{\widetilde{\Phi}_\lambda}$ can be written as

$$e^{\widetilde{\Phi}_\lambda} = \mathrm{Pexp} \left[\int_0^\lambda d\lambda' \widetilde{G}(\lambda') \right] \quad \text{with} \quad \widetilde{G}(\lambda) = H'_\lambda \frac{1}{1 - H_\lambda}. \tag{3.21}$$

It is easy to verify that $\widetilde{G}(\lambda)$ satisfies (3.1) using the following equation:

$$\frac{d}{d\lambda} \left[(Q_B H_\lambda) \frac{1}{1 - H_\lambda} - \Psi_\lambda \right] = Q_B \left(H'_\lambda \frac{1}{1 - H_\lambda} \right) + \left[\Psi_\lambda, H'_\lambda \frac{1}{1 - H_\lambda} \right] - \Psi'_\lambda = 0. \tag{3.22}$$

We can think of Φ_λ in (3.10) and $\widetilde{\Phi}_\lambda$ as different choices from solutions to (3.1).

We conclude the section by discussing the regularity conditions mentioned in the preceding section. When we proved that $G(\lambda)$ in (3.5) satisfies (3.1), we used the following relations:

$$\begin{aligned} (Q_B X)(Q_B P)(Q_B X) &= (Q_B X)(Q_B X), \\ (Q_B X)(Q_B P) X &= (Q_B X) X, \\ X (Q_B P)(Q_B X) &= X (Q_B X). \end{aligned} \tag{3.23}$$

The first two relations were discussed in [24], and they hold if $V_1(z) V_1(w)$, $V_1(z) V_{1/2}(w)$, and $V_{1/2}(z) V_{1/2}(w)$ are regular in the limit $w \to z$. The last relation also holds if these conditions are satisfied. Let us next consider if the string field $G(\lambda)$ itself is finite and if any intermediate steps in the proof are well defined. The expressions can be divergent when two or more operators collide, but if the states

$$[(Q_B X) P]^{n-1} (Q_B X), \qquad [(Q_B X) P]^{n-1} X [P (Q_B X)]^{m-1} \tag{3.24}$$

for any positive integers n and m are finite, the string field $G(\lambda)$ and any intermediate steps in the proof are well defined. The conditions for $[(Q_B X)P]^{n-1}(Q_B X)$ to be finite were discussed in [24], and it is straightforward to extend the discussion to $[(Q_B X)P]^{n-1}X[P(Q_B X)]^{m-1}$. It is easy to confirm that the bc ghost sector is finite. For the superghost sector, there is a new term of the form $\eta e^{\phi}(1)\xi e^{-\phi}(1+\ell_{n-1})\eta e^{\phi}(1+\ell_{n+m-2})$, but it is regular as well. Therefore, all the expressions are well defined if the contributions from the matter sector listed below are finite:

$$\begin{aligned} &\int_0^1 dt V_\alpha(1) V_\gamma(1+t), \\ &\int d^{n+m}t V_\alpha(1) \prod_{i=1}^{n-1}\Big[V_1(1+\ell_i)\Big] V_\beta(1+\ell_n) \prod_{j=n+1}^{n+m-1}\Big[V_1(1+\ell_j)\Big] V_\gamma(1+\ell_{n+m}) \end{aligned} \tag{3.25}$$

for any positive integers n and m, where V_α, V_β, and V_γ can be V_1 or $V_{1/2}$, and we used the notation introduced in (2.9) with the understanding that

$$\prod_{i=1}^{0}\Big[V_1(1+\ell_i)\Big] = 1, \qquad \prod_{j=n+1}^{n}\Big[V_1(1+\ell_j)\Big] = 1. \tag{3.26}$$

The only minor difference compared to the conditions for the solution in [24] is that $V_{1/2}$ can appear three times. When the string field $G(\lambda)$ is finite, the solution Φ_λ is also finite to any finite order in λ. We thus conclude that if operator products of an arbitrary number of V_1's and at most three $V_{1/2}$'s are regular, the solution Φ_λ in (3.10) made of $G(\lambda)$ in (3.5) is well defined and satisfies the equation of motion (1.1).

4. Discussion

We have constructed analytic solutions for marginal deformations satisfying the reality condition in open superstring field theory when operator products made of V_1 and $V_{1/2}$ are regular. It is important to extend the construction to the cases where the operator products are singular. Since the structure of $G(\lambda)$ is very similar to that of the solutions for the bosonic string in [20, 21], we hope that it will not be difficult to construct solutions for the superstring once we complete the program of constructing solutions with singular operator products developed in [21].[4]

It was important for the approach by Erler [23] that the equation of motion in open superstring field theory (1.1) takes the form that η_0 annihilates the pure-gauge configuration $e^{-\Phi}Q_B e^{\Phi}$ of open bosonic string field theory. Interestingly, the equation of motion in heterotic string field theory [43, 44] takes the form that η_0 annihilates a pure-gauge configuration of closed bosonic string field theory [45 – 50]. Therefore, a similar approach may be useful in constructing solutions in heterotic string field theory once we find solutions in closed bosonic string field theory.

[4]It is not clear if the recent approach to the construction of solutions with singular operator products in [42] can be directly extended to the superstring within our framework.

The open superstring field theory formulated by Berkovits can also be used to describe the $N = 2$ string by replacing Q_B and η_0 with the generators in the $N = 2$ string [22]. The reality condition for the $N = 2$ string is different from that for the ordinary superstring, and it is not clear if an approach similar to the one in this paper will be useful in constructing solutions satisfying the reality condition for the $N = 2$ string.

Open superstring field theory formulated by Berkovits [22] is more than ten years old, and its first analytic solutions have now been constructed. We expect further exciting developments in the near future.

Note added. The convention for the conjugation associated with the reality condition in this paper and in [24] is different from the one used in [23, 51, 52]. Let us explain the relation between the two conventions. The string field must have a definite parity under the combination of the Hermitean conjugation (hc) and the inverse BPZ conjugation (bpz^{-1}) to guarantee that the string field theory action is real [53]. If we denote the conjugate of a string field A in this paper and in [24] by A^*, it is defined by

$$A^* \equiv \begin{cases} \mathrm{bpz}^{-1} \circ \mathrm{hc}(A) & \text{when the ghost number of } A \text{ is 0 or 3 mod 4,} \\ -\mathrm{bpz}^{-1} \circ \mathrm{hc}(A) & \text{when the ghost number of } A \text{ is 1 or 2 mod 4.} \end{cases}$$

With this definition, the following relations hold:

$$(Q_B A)^* = Q_B A^*, \qquad (AB)^* = (-1)^{AB} B^* A^*,$$

where $(-1)^{AB} = -1$ when both A and B are Grassmann odd and $(-1)^{AB} = 1$ in other cases. If we denote the conjugate of a string field A used in [23, 51, 52] by $A^\ddagger$, it is defined by

$$A^\ddagger \equiv \mathrm{bpz}^{-1} \circ \mathrm{hc}(A)$$

for any ghost number. With this definition, the following relations hold:

$$(Q_B A)^\ddagger = -(-1)^A Q_B A^\ddagger, \qquad (AB)^\ddagger = B^\ddagger A^\ddagger,$$

where $(-1)^A = -1$ when A is Grassmann odd and $(-1)^A = 1$ when A is Grassmann even. The open superstring field Φ has ghost number 0, and thus $\Phi^\ddagger = \Phi^*$. The reality condition is satisfied when $\Phi^\ddagger = \Phi^* = -\Phi$. The open bosonic string field Ψ has ghost number 1, and thus $\Psi^\ddagger = -\Psi^*$. The reality condition is satisfied when $\Psi^\ddagger = -\Psi^* = \Psi$.

Acknowledgments

I would like to thank Ted Erler for helpful correspondence.

A. Expansion

In this appendix we present the expansion of the solution Φ_λ to third order in λ. We first expand $G(\lambda)$ in powers of X:

$$\begin{aligned} G(\lambda) = X &+ \lambda[(Q_B X)PX + XP(Q_B X)] \\ &+ \lambda^2[(Q_B X)P(Q_B X)PX + (Q_B X)PXP(Q_B X) + XP(Q_B X)P(Q_B X)] \\ &+ O(X^4). \end{aligned} \tag{A.1}$$

The expansion of e^{Φ_λ} is

$$
\begin{aligned}
e^{\Phi_\lambda} &= \mathrm{Pexp}\left[\int_0^\lambda d\lambda' G(\lambda')\right] \\
&= 1 + \int_0^\lambda d\lambda_1 G(\lambda_1) + \int_0^\lambda d\lambda_1 \int_0^{\lambda_1} d\lambda_2 G(\lambda_2) G(\lambda_1) \\
&\quad + \int_0^\lambda d\lambda_1 \int_0^{\lambda_1} d\lambda_2 \int_0^{\lambda_2} d\lambda_3 G(\lambda_3) G(\lambda_2) G(\lambda_1) + O(X^4) \\
&= 1 + \lambda X + \frac{1}{2}\lambda^2 [(Q_B X) P X + X P (Q_B X) + X X] \\
&\quad + \lambda^3 \left[\frac{1}{3}(Q_B X) P (Q_B X) P X + \frac{1}{3}(Q_B X) P X P (Q_B X) + \frac{1}{3} X P (Q_B X) P (Q_B X)\right. \\
&\qquad + \frac{1}{3} X (Q_B X) P X + \frac{1}{3} X X P (Q_B X) + \frac{1}{6}(Q_B X) P X X + \frac{1}{6} X P (Q_B X) X \\
&\qquad \left. + \frac{1}{6} X X X\right] + O(X^4).
\end{aligned}
\tag{A.2}
$$

The expansion of the solution Φ_λ is given by

$$
\begin{aligned}
\Phi^{(1)} &= X, \\
\Phi^{(2)} &= \frac{1}{2}[(Q_B X) P X + X P (Q_B X)], \\
\Phi^{(3)} &= \frac{1}{3}(Q_B X) P (Q_B X) P X + \frac{1}{3}(Q_B X) P X P (Q_B X) + \frac{1}{3} X P (Q_B X) P (Q_B X) \\
&\quad + \frac{1}{12} X (Q_B X) P X + \frac{1}{12} X X P (Q_B X) - \frac{1}{12}(Q_B X) P X X - \frac{1}{12} X P (Q_B X) X.
\end{aligned}
\tag{A.3}
$$

Note that $\Phi^{(1)}$, $\Phi^{(2)}$, and $\Phi^{(3)}$ satisfy the reality condition. The BRST transformation of Φ_λ to $O(\lambda^3)$ is given by

$$
\begin{aligned}
Q_B \Phi^{(1)} &= Q_B X, \\
Q_B \Phi^{(2)} &= (Q_B X) P (Q_B X) - \frac{1}{2}(Q_B X) X + \frac{1}{2} X (Q_B X), \\
Q_B \Phi^{(3)} &= (Q_B X) P (Q_B X) P (Q_B X) \\
&\quad + \frac{1}{2} X (Q_B X) P (Q_B X) - \frac{1}{2}(Q_B X) P (Q_B X) X \\
&\quad - \frac{1}{4}(Q_B X)[(Q_B X) P X + X P (Q_B X)] + \frac{1}{4}[(Q_B X) P X + X P (Q_B X)](Q_B X) \\
&\quad + \frac{1}{12} X X (Q_B X) - \frac{1}{6} X (Q_B X) X + \frac{1}{12}(Q_B X) X X.
\end{aligned}
\tag{A.4}
$$

Let us next expand the equation of motion. Since

$$
\begin{aligned}
e^{-\Phi} Q_B e^{\Phi} &= Q_B \Phi + \frac{1}{2}(Q_B \Phi)\Phi - \frac{1}{2}\Phi(Q_B \Phi) \\
&\quad + \frac{1}{6}(Q_B \Phi)\Phi^2 - \frac{1}{3}\Phi(Q_B \Phi)\Phi + \frac{1}{6}\Phi^2(Q_B \Phi) + O(\Phi^4),
\end{aligned}
\tag{A.5}
$$

we have

$$\eta_0 Q_B \Phi^{(1)} = 0,$$

$$\eta_0 \left[Q_B \Phi^{(2)} + \frac{1}{2}(Q_B \Phi^{(1)})\Phi^{(1)} - \frac{1}{2}\Phi^{(1)}(Q_B \Phi^{(1)}) \right] = 0,$$

$$\begin{aligned} \eta_0 \Big[& Q_B \Phi^{(3)} + \frac{1}{2}(Q_B \Phi^{(1)})\Phi^{(2)} + \frac{1}{2}(Q_B \Phi^{(2)})\Phi^{(1)} - \frac{1}{2}\Phi^{(1)}(Q_B \Phi^{(2)}) - \frac{1}{2}\Phi^{(2)}(Q_B \Phi^{(1)}) \\ & + \frac{1}{6}(Q_B \Phi^{(1)})\Phi^{(1)}\Phi^{(1)} - \frac{1}{3}\Phi^{(1)}(Q_B \Phi^{(1)})\Phi^{(1)} + \frac{1}{6}\Phi^{(1)}\Phi^{(1)}(Q_B \Phi^{(1)}) \Big] = 0. \end{aligned} \tag{A.6}$$

It is easy to confirm that $\Phi^{(1)}$, $\Phi^{(2)}$, and $\Phi^{(3)}$ in (A.3) satisfy these equations.

References

[1] E. Witten, *Noncommutative geometry and string field theory*, *Nucl. Phys.* **B 268** (1986) 253.

[2] W. Taylor and B. Zwiebach, *D-branes, tachyons and string field theory*, hep-th/0311017.

[3] A. Sen, *Tachyon dynamics in open string theory*, *Int. J. Mod. Phys.* **A 20** (2005) 5513 [hep-th/0410103].

[4] L. Rastelli, *String field theory*, hep-th/0509129.

[5] W. Taylor, *String field theory*, hep-th/0605202.

[6] M. Schnabl, *Analytic solution for tachyon condensation in open string field theory*, *Adv. Theor. Math. Phys.* **10** (2006) 433 [hep-th/0511286].

[7] Y. Okawa, *Comments on Schnabl's analytic solution for tachyon condensation in Witten's open string field theory*, *JHEP* **04** (2006) 055 [hep-th/0603159].

[8] E. Fuchs and M. Kroyter, *On the validity of the solution of string field theory*, *JHEP* **05** (2006) 006 [hep-th/0603195].

[9] E. Fuchs and M. Kroyter, *Schnabl's $\mathcal{L}_0$ operator in the continuous basis*, *JHEP* **10** (2006) 067 [hep-th/0605254].

[10] L. Rastelli and B. Zwiebach, *Solving open string field theory with special projectors*, hep-th/0606131.

[11] I. Ellwood and M. Schnabl, *Proof of vanishing cohomology at the tachyon vacuum*, *JHEP* **02** (2007) 096 [hep-th/0606142].

[12] H. Fuji, S. Nakayama and H. Suzuki, *Open string amplitudes in various gauges*, *JHEP* **01** (2007) 011 [hep-th/0609047].

[13] E. Fuchs and M. Kroyter, *Universal regularization for string field theory*, *JHEP* **02** (2007) 038 [hep-th/0610298].

[14] Y. Okawa, L. Rastelli and B. Zwiebach, *Analytic solutions for tachyon condensation with general projectors*, hep-th/0611110.

[15] M. Asano and M. Kato, *New covariant gauges in string field theory*, hep-th/0611189.

JHEP09(2007)082

[16] M. Asano and M. Kato, *Level truncated tachyon potential in various gauges*, *JHEP* **01** (2007) 028 [hep-th/0611190].

[17] T. Erler, *Split string formalism and the closed string vacuum*, *JHEP* **05** (2007) 083 [hep-th/0611200].

[18] C. Imbimbo, *The spectrum of open string field theory at the stable tachyonic vacuum*, *Nucl. Phys.* **B 770** (2007) 155 [hep-th/0611343].

[19] T. Erler, *Split string formalism and the closed string vacuum. II*, *JHEP* **05** (2007) 084 [hep-th/0612050].

[20] M. Schnabl, *Comments on marginal deformations in open string field theory*, hep-th/0701248.

[21] M. Kiermaier, Y. Okawa, L. Rastelli and B. Zwiebach, *Analytic solutions for marginal deformations in open string field theory*, hep-th/0701249.

[22] N. Berkovits, *Super Poincaré invariant superstring field theory*, *Nucl. Phys.* **B 450** (1995) 90 [*Erratum ibid.* **B 459** (1996) 439] [hep-th/9503099].

[23] T. Erler, *Marginal solutions for the superstring*, *JHEP* **07** (2007) 050 [arXiv:0704.0930].

[24] Y. Okawa, *Analytic solutions for marginal deformations in open superstring field theory*, arXiv:0704.0936.

[25] A. Sen and B. Zwiebach, *Large marginal deformations in string field theory*, *JHEP* **10** (2000) 009 [hep-th/0007153].

[26] A. Iqbal and A. Naqvi, *On marginal deformations in superstring field theory*, *JHEP* **01** (2001) 040 [hep-th/0008127].

[27] T. Takahashi and S. Tanimoto, *Wilson lines and classical solutions in cubic open string field theory*, *Prog. Theor. Phys.* **106** (2001) 863 [hep-th/0107046].

[28] M. Mariño and R. Schiappa, *Towards vacuum superstring field theory: the supersliver*, *J. Math. Phys.* **44** (2003) 156 [hep-th/0112231].

[29] J. Kluson, *Exact solutions of open bosonic string field theory*, *JHEP* **04** (2002) 043 [hep-th/0202045].

[30] T. Takahashi and S. Tanimoto, *Marginal and scalar solutions in cubic open string field theory*, *JHEP* **03** (2002) 033 [hep-th/0202133].

[31] J. Kluson, *Marginal deformations in the open bosonic string field theory for N D0-branes*, *Class. and Quant. Grav.* **20** (2003) 827 [hep-th/0203089].

[32] J. Kluson, *Exact solutions in open bosonic string field theory and marginal deformation in CFT*, *Int. J. Mod. Phys.* **A 19** (2004) 4695 [hep-th/0209255].

[33] J. Kluson, *Exact solutions in SFT and marginal deformation in BCFT*, *JHEP* **12** (2003) 050 [hep-th/0303199].

[34] E. Coletti, I. Sigalov and W. Taylor, *Abelian and nonabelian vector field effective actions from string field theory*, *JHEP* **09** (2003) 050 [hep-th/0306041].

[35] N. Berkovits and M. Schnabl, *Yang-Mills action from open superstring field theory*, *JHEP* **09** (2003) 022 [hep-th/0307019].

JHEP09(2007)082

[36] A. Sen, *Energy momentum tensor and marginal deformations in open string field theory*, *JHEP* **08** (2004) 034 [hep-th/0403200].

[37] F. Katsumata, T. Takahashi and S. Zeze, *Marginal deformations and closed string couplings in open string field theory*, *JHEP* **11** (2004) 050 [hep-th/0409249].

[38] H. Yang and B. Zwiebach, *Testing closed string field theory with marginal fields*, *JHEP* **06** (2005) 038 [hep-th/0501142].

[39] I. Kishimoto and T. Takahashi, *Marginal deformations and classical solutions in open superstring field theory*, *JHEP* **11** (2005) 051 [hep-th/0506240].

[40] D. Friedan, E.J. Martinec and S.H. Shenker, *Conformal invariance, supersymmetry and string theory*, *Nucl. Phys.* **B 271** (1986) 93.

[41] J. Polchinski, *String theory Vol. 2: Superstring theory and beyond*, Cambridge University Press, Cambridge U.K. (1998).

[42] E. Fuchs, M. Kroyter and R. Potting, *Marginal deformations in string field theory*, arXiv:0704.2222.

[43] Y. Okawa and B. Zwiebach, *Heterotic string field theory*, *JHEP* **07** (2004) 042 [hep-th/0406212].

[44] N. Berkovits, Y. Okawa and B. Zwiebach, *WZW-like action for heterotic string field theory*, *JHEP* **11** (2004) 038 [hep-th/0409018].

[45] B. Zwiebach, *Closed string field theory: quantum action and the B-V master equation*, *Nucl. Phys.* **B 390** (1993) 33 [hep-th/9206084].

[46] M. Saadi and B. Zwiebach, *Closed string field theory from polyhedra*, *Ann. Phys. (NY)* **192** (1989) 213.

[47] T. Kugo, H. Kunitomo and K. Suehiro, *Nonpolynomial closed string field theory*, *Phys. Lett.* **B 226** (1989) 48.

[48] T. Kugo and K. Suehiro, *Nonpolynomial closed string field theory: action and its gauge invariance*, *Nucl. Phys.* **B 337** (1990) 434.

[49] M. Kaku, *Geometric derivation of string field theory from first principles: closed strings and modular invariance*, *Phys. Rev.* **D 38** (1988) 3052.

[50] M. Kaku and J.D. Lykken, *Modular invariant closed string field theory*, *Phys. Rev.* **D 38** (1988) 3067.

[51] M. Kiermaier and Y. Okawa, *Exact marginality in open string field theory: a general framework*, arXiv:0707.4472.

[52] M. Kiermaier and Y. Okawa, *General marginal deformations in open superstring field theory*, arXiv:0708.3394.

[53] M.R. Gaberdiel and B. Zwiebach, *Tensor constructions of open string theories I: foundations*, *Nucl. Phys.* **B 505** (1997) 569 [hep-th/9705038].

JHEP09(2007)082